

PLANNING MARKETING-MIX STRATEGIES IN THE PRESENCE OF INTERACTION EFFECTS:
EMPIRICAL AND EQUILIBRIUM ANALYSES

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ABSTRACT

Companies spend millions of dollars on advertising to boost a brand's image, while simultaneously spending millions of dollars on promotion that calls attention to price and erodes brand equity. This situation arises because both advertising and promotion are necessary to compete effectively in dynamic markets. Consequently, brand managers need to account for interactions between marketing activities and interactions among competing brands in determining the appropriate level of budget and its allocation to marketing activities. By recognizing interaction effects between activities, managers can consider inter-activity tradeoffs in planning the marketing-mix strategies. On the other hand, by recognizing interactions with competitors, managers can incorporate strategic foresight in their planning, which requires them to look forward and reason backwards in making optimal decisions. Looking forward means that each brand manager anticipates how other competing brands are likely to make future decisions and then, by reasoning backwards, deduces one's own optimal decisions in response to the best decisions to be made by all other brands. The joint consideration of interaction effects and strategic foresight in planning marketing-mix strategies is a challenging and open marketing problem, which motivates this paper.

In this paper, we develop a methodology for planning optimal marketing-mix in dynamic competitive markets, taking into account strategic foresight and interaction effects. To estimate and infer the existence of interaction effects, we design a Kalman filter that uses readily available discrete-time market data to calibrate continuous-time marketing models of dynamic competition. To develop optimal marketing-mix plans, we construct a computational algorithm for solving the nonlinear two-point boundary value problem associated with the derivation of equilibrium strategies. We illustrate the application of this dual methodology by studying the dynamic Lanchester competition across five brands in the detergents markets, where each brand uses advertising and promotion to influence its own market share and the shares of competing brands. Empirically, we find that advertising and promotion not only affect the brand shares (own and competitors'), but also exert interaction effects, i.e., each activity amplifies or attenuates the effectiveness of the other activity. Moreover, if managers ignore these interaction effects, they are likely to believe that advertising and promotion are less effective than they actually are. Normatively, we find that large brands such as Tide and Wisk not only under-advertise, but also substantially over-spend on promotion. Thus, we provide support for the view that "escalation of promotion" may exist in some markets, a finding that Leeflang and Wittink (2001) attribute to managers' lack of strategic foresight. Finally, the generality of both the estimation method and computational algorithm enables managers to apply the proposed methodology to other market response models that reflect the marketing environment specific to their companies.

(Continuous-Discrete Estimation; Dynamic Competition; Interaction Effects; Marketing-Mix Planning; Strategic Foresight; Two-point Boundary Value Problem)

1. INTRODUCTION

American corporations collectively spend hundreds of billions of dollars on advertising and promotion, an amount that exceeds the gross domestic product of several nations such as Singapore, Sweden and Switzerland. Even individual companies, for example, Procter and Gamble, spend several billion dollars on marketing activities. Consequently, the determination of the marketing budget and its allocation to several marketing activities — referred to as “planning the marketing-mix” — is of paramount importance (see Mantrala 2002 for literature review). Recent dynamic marketing models such as Naik, Mantrala and Sawyer (1998) and Silva-Risso, Bucklin and Morrison (1999) provide decision-support tools for planning advertising schedules and promotional calendars, respectively. However, these planning models ignore the fundamental game-theoretic principle of strategic foresight, which requires managers to look forward and reason backwards in making optimal decisions. Leeflang and Wittink (2001, p. 120) argue that if managers were to incorporate strategic foresight in their planning, then “...the escalation in promotional expenditures in the US ...would not have occurred.”

The notion of strategic foresight means that each brand manager looks forward, i.e., anticipates how other competing brands are likely to make future decisions and the resulting profit consequences, and then reasons backward, i.e., deduces one’s own optimal decisions in response to the best decisions to be made by all other brands. By incorporating strategic foresight in the classic Lanchester model, Chintagunta and Vilcassim (1994) and Fruchter and Kalish (1998) investigate how firms should plan marketing investments in competitive markets. Specifically, Chintagunta and Vilcassim (1994) apply differential-game theory to determine optimal spending on marketing activities (e.g., advertising, promotion) in a duopoly. Extending Chintagunta and Vilcassim’s (1994) work, Fruchter and Kalish (1998) develop novel methods to analyze a differential-game across

multiple firms (i.e., oligopolistic competition). However, both these studies ignore the role of interaction effects among multiple activities for all brands.

The role of interaction effects manifests itself in the concept of marketing-mix. This concept is central to marketing, and it “...emphasizes that marketing efforts create sales synergistically rather than independently” (Gatignon and Hanssens 1987, p. 247). Several marketing studies have demonstrated its importance (e.g., Kuehn 1962, Parsons 1974, Swinyard and Ray 1977, Gatignon and Hanssens 1987), and this body of research captures the quintessential aspect of the mix via interactions between marketing variables (see Gatignon 1993 for literature review).

The joint consideration of these two notions — strategic foresight and interaction effects — in dynamic response models is admittedly challenging, and represents an important gap in the extant marketing literature. For example, Fruchter and Kalish (1998, p. 22) acknowledge

“...the limitations of current studies [not] to take into account the interactions among the different instruments. A challenge which we see for a future direction is to develop a model which incorporates interactions between promotional instruments.”

The challenging problems arise mainly because of the following two reasons. First, as we will show later, in the absence of interactions, the optimal plan for a marketing activity does not depend directly on the plans for other activities. In contrast, when interaction effects are present, the optimal plans for all activities are directly inter-dependent. These inter-dependencies indicate that managers should account for inter-activity tradeoffs in allocating the marketing budget. In other words, how much they spend on advertising depends on how much they plan to spend on promotion. Second, to enable the managers to determine marketing-mix plans specific to their market conditions, we require a methodology for calibrating dynamic models for oligopoly markets in the presence of interaction effects using market data. Thus, in the presence of interaction effects, both the substantive problems — the estimation of dynamic models for

oligopoly markets, and the determination of optimal budgeting and allocation strategies in equilibrium across all competing brands — are open research topics.

We address this need by developing an estimation method and computational algorithm for planning marketing-mix strategies in equilibrium for dynamic competitive markets. We first extend the classic Lanchester model by introducing the following phenomena: multiple brands in competition, multiple marketing activities that each brand employs to influence its own and other brands' shares, diminishing or increasing returns to marketing activities, and interactions between marketing activities. We then develop appropriate methods to conduct the empirical and equilibrium analyses of the proposed model extension. Specifically, applying Kalman filtering theory (see Lewis 1986), we present an estimation method to calibrate continuous-time dynamic models using discrete-time observed data. Using data for detergents market consisting of five brands, we estimate the proposed model and various alternative models. We determine the model specification to retain for further decision-making purposes based on model selection principles (see Burnham and Anderson 1998). Next, to solve the differential-game implied by the retained model, we invoke the theory of nonlinear two-point boundary values (see Ascher, Matheij, and Russell 1995), and construct a computational algorithm that yields the optimal budget and allocation plans. We derive the open-loop Nash equilibrium strategies because the planning process, by its very nature, uses all information available only at the time of planning. (We note that the computational algorithm also yields closed-loop strategies — see section 5.3.) These plans possess the desirable properties of being in equilibrium across multiple brands, and consider the inter-temporal (i.e., spending now versus later) and inter-activity (e.g., spending on advertising versus promotion) tradeoffs.

Conceptually, and perhaps most importantly, we distinguish between the use of multiple marketing activities, i.e., main effects, and the notion of the marketing-mix embodying the role of interactions between such activities (see Gatignon 1993). We demonstrate that this distinction is important because it leads to substantive differences in optimal behavior and hence managers should plan their marketing-mix differently depending on the presence or absence of such interaction effects.

To advance this conceptual distinction and obtain substantive results, we make the following two methodological contributions. First, we note that time-series models in marketing apply discrete-time estimation methods (e.g., ARIMA; see Blattberg and Neslin 1990, Ch. 9) even though dynamic models are specified in continuous-time (e.g., Lanchester model). A notable exception is Rao (1986), who studies the estimation of univariate continuous-time models. We present an approach to estimate multivariate continuous-time models using discrete-time observed data. In addition, our estimation approach takes into account the complexities introduced by interactions and interdependencies across multiple brands (see Gatignon 1993).

Our second methodological contribution is a computational algorithm that yields the normative strategies for multiple marketing activities in the presence of interaction effects for all brands. We illustrate this algorithm by planning equilibrium marketing-mix strategies for five detergent brands where each brand spends on advertising and promotion. Because we explicitly include the brand prices in the model, the optimal promotion plans belong to the class of “bang-bang” policies (i.e., promotion is either on or off). In other words, brand managers must decide whether to promote in a given week of the planning year or not. Each brand faces 2^{52} counterfactual scenarios, which represent hundreds of trillions of promotional possibilities to promote a brand in the planning year. In practice, enumeration of trillions of promotion plans is infeasible.

However, our proposed algorithm not only finds the optimal marketing-mix plans for all brands, but also incorporates strategic foresight. That is, it anticipates the expected plans of other brands and then deduces the best set of promotional plans via backward induction. Thus, the resulting plans are “optimal” in the sense of Nash equilibrium both over time and across brands.

2. MODEL DEVELOPMENT

2.1 Lanchester Model

Vidale and Wolfe (1957) developed a monopoly model of sales growth due to advertising, and its extension to a market with two competing brands yields the Lanchester model. Specifically, for brands 1 and 2, we denote market shares at time t by $m_1(t)$ and $m_2(t)$ and advertising efforts by $u_1(t)$ and $u_2(t)$ to formulate the Lanchester model:

$$(1) \quad \frac{dm_1}{dt} = \beta_1 u_1 (1 - m_1) - \beta_2 u_2 m_1.$$

In equation (1), the term dm_1/dt represents the net gain in market share of the first brand. The first term on the right-hand side indicates that the first brand gains the market share from the other brand when it increases the level or effectiveness of its advertising (i.e., when brand 1 increases u_1 or β_1). The second term on the right-hand side indicates that the first brand loses share when the second brand increases the level or effectiveness of advertising (i.e., when brand 2 increases u_2 or β_2). Similarly, formulating the dynamics of second brand’s share, $dm_2/dt = \beta_2 u_2 (1 - m_2) - \beta_1 u_1 m_2$, we observe that the Lanchester model captures the interdependence in the evolution of brand shares due to own and other brands’ advertising efforts. An important property of this model is that it ensures that brand shares always sum to unity (i.e., $dm_1/dt + dm_2/dt = 0$ for all t).

2.2 Extended Lanchester Model: Multiple Brands, Multiple Activities, and Interaction Effects

2.2.1 Multiple Brands

Consider an oligopoly with N brands. For each brand i , $i = 1, \dots, N$, we let m_i denote its market share, and f_i be the force of its marketing activities. For example, $f_1 = \beta_1 u_1$ and $f_2 = \beta_2 u_2$ in the equation (1). Then, generalizing equation (1), the market share of brand 1 evolves as follows:

$$(2) \quad \frac{dm_1}{dt} = f_1(1 - m_1) - f_2 m_1 - f_3 m_1 \cdots - f_i m_1 \cdots - f_N m_1.$$

Equation (2) says that brand 1's marketing force f_1 impacts all the other brands' share, $(1 - m_1)$, to drive its own growth, and it loses share because the marketing forces f_i of the other brands ($i = 2, 3, \dots, N$) impacts its market share m_1 . We simplify (2) to obtain $\dot{m}_1 = f_1 - F m_1$, and then generalize it to N -brand market, representing the extended Lanchester model in the vector differential form:

$$(3) \quad \begin{bmatrix} \dot{m}_1 \\ \vdots \\ \dot{m}_N \end{bmatrix} = \begin{bmatrix} f_1 \\ \vdots \\ f_N \end{bmatrix} - \begin{bmatrix} F & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & F \end{bmatrix} \begin{bmatrix} m_1 \\ \vdots \\ m_N \end{bmatrix},$$

where $\dot{m}_i = dm_i/dt$ is the time-derivative of $m_i(t)$, and $F = \sum_{i=1}^N f_i$ denotes the marketing force of all

the brands (including the first brand). Equation (3) is the N -brand Lanchester oligopoly model.

Similar to the duopoly case, it also possesses the property that brand shares sum to unity at every

instant t . To verify this, add all equations in (3) and observe that $\sum_i \dot{m}_i = \sum_i f_i - F(\sum_i m_i) =$

$F - F(\sum_i m_i) = 0$ implies $\sum_i m_i(t) = 1$ at every instant t for all nontrivial F .

2.2.2 Multiple Activities

The marketing force for brand i is generated by its marketing activities such as advertising and promotion. In particular, denoting advertising and promotional efforts by (u_i, v_i) , we specify the marketing force of brand i , $i = 1, \dots, N$,

$$(4) \quad f_i = \alpha_i u_i + \beta_i v_i,$$

where (α_i, β_i) represent the effectiveness of advertising and promotion activities. Equation (4) says that the magnitude of marketing force is specific to each brand i . It depends not only on the levels of advertising and promotional effort (u_i, v_i) , but also on the effectiveness (α_i, β_i) . In addition, both the levels of effort and the effectiveness coefficients are specific to each brand i . Finally, we focus on two marketing activities to gain expositional clarity by minimizing the notational clutter that would arise otherwise, and note that generalization of this model to several activities entails no new conceptual issues.

2.2.3 *Interaction Effects*

As mentioned in the Introduction, we make a conceptual distinction between multiple marketing activities with main effects (as in equation 6), and the interaction effects among these activities. Specifically, when managers employ multiple activities to influence their own and other brands' shares, the effectiveness of each activity amplifies or attenuates the effectiveness of other activities. Indeed, the marketing-mix concept embodies the notion that each activity moderates the effectiveness of other activities (see Gatignon and Hanssens 1987, Gatignon 1993). Hence, we introduce interaction terms in equation (4) to obtain,

$$(5) \quad f_i = \alpha_i u_i + \beta_i v_i + \gamma_i u_i v_i,$$

thus augmenting the scope of the marketing force of each brand i . Equation (5) says that, when brand i expends the effort u_i , this activity not only increases its brand share m_i , but also influences the effectiveness of the other activity v_i because of the non-zero γ . When $\gamma > 0$, the two marketing activities interact positively, thus creating synergy, which strengthens the marketing force. When $\gamma < 0$, the two activities interact negatively, which weakens the marketing force. In either case, the interaction term captures the non-linear effect of marketing activities, reflecting the joint impact of

activities in addition to the main effects of each activity (see Gatignon and Hanssens 1987 for further details).

Despite its simplicity, equation (5) can incorporate additional non-linear phenomena such as saturation and threshold effects (Simon and Arndt 1980, Van Heerde, Leeflang, and Wittink 2001). To achieve this goal, we could transform $u_i = \sqrt{x_i}$ or $u_i = \ln(1 + x_i)$ to capture diminishing returns to the marketing activity measured by x_i . Similarly, we represent increasing returns via the transformation $u_i = x_i^2$ or $u_i = \exp(x_i)$.

The extension from equation (4) to equation (5), while deceptively simple in appearance, raises a host of challenges in both the empirical and equilibrium analyses (see Gatignon 1993, Fruchter and Kalish 1998). Specifically, estimation of the resulting dynamic model requires the theory of continuous-discrete Kalman filtering (see Lewis 1986), and the derivation of the equilibrium strategies utilizes the methodology of nonlinear two-point boundary value problems (see Ascher, Matheij, and Russell 1995). These dual technologies enable managers to (a) quantify the magnitudes of synergy or negative interaction for each brand i , (b) infer their statistical significance, and (c) obtain equilibrium plans for budgeting and allocation across all brands in dynamic oligopolies in the presence interactions. In the next section, we develop the necessary tools for the empirical analysis.

3. EMPIRICAL ANALYSIS

3.1 Data

We use single-source data from the detergents market to study dynamic competition for market shares among the five brands: Wisk, Tide, Bold, Era, and Solo. The data set consists of the purchase histories of households, the prices paid, the cents-off deals received, and the TV

viewing behavior over a period of 84 weeks (see Winer 1993 for details). Because the Lanchester model specifies the unit of analysis at the brand level (and not at the household level), we aggregate the household purchases to obtain the brand shares over time. Specifically, we operationalize the variables for advertising, promotion, and sales as follows.

Advertising. We identify the households who bought a brand in week k , and denote this set of consumers by $C_k = \{c_1, \dots, c_h, \dots, c_H\}$. For each consumer c_h , we first determine the time when s/he made the last purchase. During this inter-purchase interval, which is specific to each consumer, we compute the ad time τ_{ihk} devoted by the consumer h to watch advertisements for brands $i = 1, \dots, 5$. We then sum the ad time across all the households in the set C_k . That is, we compute the measure for “advertising” by brand i in week k as $\tilde{u}_{ik} = \sum_{c_h \in C_k} \tau_{ihk}$. It represents the total number of seconds of television ads watched by those specific households who bought the brand in a given week. In the estimation, to make measurements dimensionless, i.e., unit-free, we use the standardized measure x_{1ik} obtained by dividing \tilde{u}_{ik} with its standard deviation; that is, $x_{1ik} = \tilde{u}_{ik} / \sigma_{\tilde{u}_i}$.

Price Promotion. The brand-level information on price and promotion \tilde{p}_{ik} and \tilde{v}_{ik} , respectively, is directly available for every week. Consistent with the advertising measure, we standardize \tilde{p}_{ik} and \tilde{v}_{ik} by dividing them with their corresponding standard deviations; that is, $p_{ik} = \tilde{p}_{ik} / \sigma_{\tilde{p}_i}$ and $x_{2ik} = \tilde{v}_{ik} / \sigma_{\tilde{v}_i}$. In our preliminary analyses, we assessed the robustness of the estimation results by operationalizing promotion in three different ways: (a) cents-off, (b) percentage price-cut, and (c) presence-absence dummy (i.e., promotion signal). We use the cents-off measure because the findings are qualitatively similar across these measures.

Brand Shares. Denoting the units bought by the consumer h by q_{hik} , we determine sales for each brand i in week k by summing over the households in set C_k ; that is, $s_{ik} = \sum_{c_h \in C_k} q_{hik}$.

Using this information, we construct the brand shares $y_{ik} = \frac{s_{ik}}{\sum_i s_{ik}} \times 100$ for $i = 1, \dots, 5$. We do not standardize brand shares because they are unit-free by construction. For example, the average percentage shares are as follows: Tide (43.37), Wisk (19.20), Era (15.45), Solo (12.69), and Bold (9.29).

The two advantages of this data and model are as follows. First, unlike aggregate data, the single-source data allow us to match the purchase and TV viewing behavior of households. In other words, the above operationalization ensures that consumers who generated “sales” are exactly the same households who watched the advertisements. In the previous studies, by contrast, consumers who buy the brands and the viewers who watch the advertisements are two different sets of households. Second, unlike disaggregate models, the Lanchester model is empirically applicable to several competitive markets such as apparel (e.g., Levi’s, Gap), cars (e.g., Camry, Maxima), computers (e.g., Dell, Gateway), chips (e.g., Pentium, Athalon), phones (e.g., Nokia, Motorola) and many other products and services. This is because the information on sales and marketing activities for these brands exists at the brand-level, but not at the household-level. Next, we develop an estimation approach to use the readily available aggregate data for calibrating the extended Lanchester model.

3.2 Estimation Approach

3.2.1 Continuous-Discrete Kalman Filtering

We note that brand-share dynamics specified by equation (3) are in continuous-time. That is, in the model, the time parameter t is continuously differentiable on any interval,

$\underline{t} < t < \bar{t}$. However, brand-share data arrive at discrete points in time (e.g., weeks). That is, in the data series, the time parameter t is not continuously differentiable; rather it takes discrete values in the integer set $\{1, 2, 3, \dots, T\}$. Rao (1986) investigates the resulting biases under these conditions for univariate data. We extend this approach to calibrate multivariate continuous-time models using discrete-time observed data (Lewis 1986, p. 185).

To explain the approach, let m_k denote the $N \times 1$ vector of brand shares at time t_k , where k is an integer taking values in the index set $\{1, 2, \dots, T\}$. Using equation (3), we integrate over the interval $(t_{k-1}, t_k]$ to obtain the exact equation for relating brand shares at two discrete points in time:

$$m_k - m_{k-1} = \int_{t_{k-1}}^{t_k} dm = \int_{t_{k-1}}^{t_k} \left(\frac{dm}{dt} \right) dt = \int_{t_{k-1}}^{t_k} (\dot{m}) dt,$$

which, upon simplification, yields the matrix transition equation,

$$(6) \quad m_k = \Phi_k m_{k-1} + \delta_k,$$

where $\Phi_k = \exp(A_k)$ is a $N \times N$ transition matrix, $\exp(\cdot)$ is the matrix exponentiation function, A_k is a diagonal $N \times N$ matrix with $(-F_k)$ as the elements on the principal diagonal, and the $N \times 1$ drift vector $\delta_k = (\exp(A_k) - I_N)A_k^{-1}f_k$, where I_N is an N -dimensional identity matrix, and $f_k = (f_{1k}, \dots, f_{Nk})'$ is the vector of marketing forces for the N brands.

Next, let y_{ik} denote the observed data on brand shares at time point t_k . Since brand shares sum to 100%, once we know $(N-1)$ brand shares, $(y_{1k}, \dots, y_{N-1,k})'$, the realized share for the

remaining brand, y_{Nk} , provides no new information, because it must equal $100 - \sum_{i=1}^{N-1} y_{ik}$.

Consequently, we only need information on any $(N-1)$ brands, and we link those $(N-1)$ equations in (6) to the observed brand shares via the observation equation,

$$(7) \quad \begin{bmatrix} y_{1k} \\ \vdots \\ y_{N-1,k} \end{bmatrix} = \begin{bmatrix} 100 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 100 \end{bmatrix} \begin{bmatrix} m_{1k} \\ \vdots \\ m_{N-1,k} \end{bmatrix},$$

where $Y_k = (y_{1k}, \dots, y_{N-1,k})'$ are observed brand shares (in percentages), and the diagonal entries convert the fractional shares to their percentage values.

We now cast the model equations (6) and (7) into the state-space form (see, e.g., Shumway and Stoffer 2000),

$$(8) \quad \begin{aligned} Y_k &= z m_k + \varepsilon_k, \\ m_k &= \Phi_k m_{k-1} + \delta_k + v_k, \end{aligned}$$

where z is a diagonal matrix with its principal elements equal to 100, and the error-terms follow the normal distributions, $\varepsilon_k \sim N(0, H)$ and $v_k \sim N(0, Q)$, where H is the diagonal matrix $\sigma^2 I_N$, and Q is the diagonal matrix, $\text{diag}(\sigma_1^2, \dots, \sigma_N^2)$. The error-terms serve as perturbations to represent the joint effect of myriad factors not explicitly included in the model, and ensure that the resulting estimators possess desirable properties (e.g., asymptotically unbiased, minimum variance). Applying the Kalman filter recursions, we then compute the log-likelihood function, maximize it with respect to the parameter vector to obtain both the maximum-likelihood estimates and the standard errors (see Shumway and Stoffer 2000 for details).

3.2.2 Sources of Bias in Estimating Multivariate Dynamic Systems

The ordinary least squares (OLS) estimation of the multivariate dynamic system in (8) results in biased estimates for the following three reasons. First, the log-likelihood function implied by the OLS regression is misspecified because it is a product of the marginal densities $f(Y_k)$, which assumes independence between brand shares at any two points in time, Y_k and $Y_{k'}$. Clearly, such temporal independence cannot exist because the differential equation (3) explicitly

induces dependence over time. Second, this temporal dependence does not vanish (or diminish over time) because brand-share dynamics in equation (6) are non-stationary. This fact is evident from the transition matrix Φ_k , which is time varying because it depends on the period k . Finally, all N brands in the system of equations (6) are highly inter-dependent on each other's marketing activities. Specifically, both the transition matrix Φ_k and the drift vector δ_k depend non-linearly on the total and individual marketing forces F and f_i (see the text after equation 8), which in turn, depend non-linearly on the marketing activities (via equation 7).

Hence, to calibrate the simultaneous system of dynamic equations (8), we should take into account the (a) temporal dependence in brand shares, (b) non-stationary dynamics of brand shares, and (c) inter-dependencies due to marketing activities. We construct the likelihood function by accounting for these factors in the proposed continuous-discrete estimation approach. We apply this approach to the detergents market, and obtain the following empirical results.

3.3 Empirical Results

3.3.1 Model Fit and Cross-validation

The proposed model of dynamic competitive market, given by (8), fits this data set quite well. The fit is uniformly satisfactory across the five brands of detergents (e.g., R^2 values in high 90s). We conduct a cross-validation study to assess the predictive validity of this model. Specifically, we use 72 observations for model calibration, and keep 12 observations as the holdout sample to assess the out-of-sample performance. We find high R^2 values for out-of-sample forecasts across all five brands. It is possible that R^2 measure may not be diagnostic for highly nonlinear models, and so we conduct model comparison and selection using information criteria (e.g., AIC).

[Insert Table 1 here]

3.3.2 Advertising and Promotion Effectiveness

Table 1 presents the standardized coefficients and t-values for the model parameters. We can compare their magnitudes because the standardized coefficients are dimensionless (i.e., unit-free). We note that, except for Bold, ad effectiveness of all other brands is statistically significant at the 95% confidence level. Ad effectiveness varies considerably in this detergents market. Specifically, Tide's advertising is the most effective, Wisk's advertising is half as effective as Tide's, and Solo's advertising is the least effective.

Promotion effectiveness of all brands, except Bold, is statistically significant at the 95% confidence level, whereas Bold's promotion effectiveness is significant at the 90% confidence level. Tide's promotion is the most effective, while Wisk's promotion is the least effective. The promotion effectiveness parameters of Era, Bold, and Solo are approximately equal. Thus, relative to ad effectiveness, promotional effectiveness displays less variability in this detergents market.

3.3.3 Interaction Effects

Table 1 shows that all brands, except Era, indicate the presence of negative interactions between advertising and promotion activities. The individual estimate of negative interaction for Tide is large and significant at the 95% confidence level. The individual estimates of negative interactions for Wisk, Solo and Bold are small and insignificant. For Era, we find evidence of synergy — rather than negative interaction — between advertising and promotion, but this estimate is not significant at the 95% confidence level. For assessing the joint significance, we test the hypothesis that all interaction effects are zero. Specifically, we test the null $H_0: \gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = \gamma_5 = 0$ using the likelihood ratio test. We note that the log-likelihood value, L_1 , for the proposed model equals -852.55 (see Panel A, Table 1), whereas that for the nested model is $L_2 =$

-862.85 (Panel B, Table 1). Consequently, the LR statistic = $-2(L_2 - L_1) = 20.6$, which exceeds the critical $\chi^2 = 11.1$ for 5 degrees of freedom at the 95% confidence level. Thus, we reject the null hypothesis of no interaction effects.

We further ascertain the validity of this conclusion using model selection principles (see Burnham and Anderson 1998) to select a model with or without interactions. We compare the corresponding values for information criteria (AIC and AIC_C) across Panels A and B of Table 1, and find that the proposed model yields smaller values on both the criteria, thus suggesting the retention of the proposed model with interaction effects. Based on model selection and hypothesis testing, we conclude that interactions exist in this dynamic competitive detergents market. Moreover, the proposed estimation approach provides brand managers a tool to determine not only the effectiveness of advertising and promotion activities, but also the magnitude and significance of interactions between them.

3.3.4 Consequences and Interpretation of Interaction Effects

To understand the consequences of ignoring interactions when they do exist, we compare the corresponding standardized coefficients for ad and promotion effectiveness in Panels A and B of Tables 1. We observe that both ad and promotion effectiveness are systematically under-estimated. This finding holds true for all five brands. For ad effectiveness, the under-estimation varies between a minimum of 17.6% to a maximum of 62.1%. For promotion effectiveness, the under-estimation ranges from 13.0% to 39.4%. Thus, when managers ignore the interaction effects, they are likely to believe that marketing activities are substantially less effective than they actually are.

One interpretation of negative interaction effect, as in Mela et al. (1997) and Jedidi et al. (1999), is that “promotions are bad” — but not because they directly hurt brand shares. Indeed, the direct impact of promotions, $\hat{\beta}_i$, $i = 1, \dots, 5$, is significantly positive, making it a powerful

marketing activity. Rather, promotions negatively moderate the impact of advertising, reducing the effectiveness of advertising in building brands.

An alternative interpretation for negative interaction between advertising and promotion is that advertising lowers promotion effectiveness.¹ It is possible that effective brand-equity focused advertising reduces the effectiveness of price promotions. While this does not seem as likely as the first explanation, the model and the empirical results only indicate the presence of a negative joint impact of advertising and promotion in a dynamic oligopoly, but not the source of interaction effects. (This situation holds for all models of interaction effects, not just this one.) It is important to recognize, however, that the normative analysis in the next section appropriately incorporates both the signs and magnitudes of all interactions in determining the equilibrium marketing-mix plans.

3.4 Model Selection

To explore alternative model specifications,² we specify and estimate five different models for dynamic oligopolistic competition (in addition to the above two models, with and without interactions). Specifically, we include the price effects of the five brands, and test for diminishing- and increasing-returns to advertising and promotion. To retain the best model, we apply model selection principles, which we briefly explain.

3.4.1 Model selection principle

The central idea of model selection is that one can add variables (i.e., include more phenomena) in the model to improve the model's fit to the observed data, but the resulting over-parameterization hurts not only the precision of parameter estimates but also the accuracy of model

¹ We thank the AE and a reviewer for suggesting this point.

² We thank the AE and reviewers for suggesting these analyses.

forecasts (e.g., Altham 1984). Hence, to maintain a proper balance between fit and parsimony, we estimate the expected Kullback-Leibler distance between the true and a tentative model, and refer to it as the information criterion (see Burnham and Anderson 1998 for details). The smaller the value of information criterion, the closer the tentative model is to the unknown true model. Specifically, we compute Akaike's information criterion, $AIC = -2L + 2p$, where L is the maximized log-likelihood value, and p is the number of model parameters. We also apply the small sample bias-corrected version, $AIC_C = -2L + T(T + p)/(T - p - 2)$, where T is the sample size (see Hurvich and Tsai 1989). Next, we specify the alternative model specifications, compute these criteria, and retain the model specification that yields the smallest value.

3.4.2 Price Effects with Interaction Effects

A brand's market share can be affected by one's own price level as well as prices of other competing brands. To incorporate own- and cross-brand price effects, we augment the equation (7) as follows:

$$(9) \quad \begin{bmatrix} y_{1k} \\ \vdots \\ y_{4k} \end{bmatrix} = \begin{bmatrix} 100 & & & \\ & \ddots & & \\ & & 100 & \\ & & & \ddots \end{bmatrix} \begin{bmatrix} m_{1k} \\ \vdots \\ m_{4k} \end{bmatrix} + \begin{bmatrix} \phi_{11} & \cdots & \phi_{15} \\ \vdots & \ddots & \vdots \\ \phi_{41} & \cdots & \phi_{45} \end{bmatrix} \begin{bmatrix} p_{1k} \\ \vdots \\ p_{5k} \end{bmatrix}.$$

In equation (9), p_{ik} denotes the price of brand i in period k ; the coefficients ϕ_{ii} represent the effect of brand i 's price on its own market share; and ϕ_{ij} ($i \neq j$) denotes the cross-price effect of brand j 's price on brand i 's share.

Table 2 presents the information criteria for alternative models of dynamic oligopoly. For the price-effects model (9), the values for $AIC = 1816.01$ and $AIC_C = 1984.01$. The corresponding values for the dynamic model with interactions are 1745.01 and 1845.99. Because $1745.01 < 1816.01$ and $1845.99 < 1984.01$, both the information criteria concur that the dynamic model with interactions is better than the price-effects model (9).

[Insert Table 2 here]

3.4.3 *Price Effects without Interactions*

To compare the price-effects model but without interaction effects, we re-estimate the model (9) with the interaction terms $\gamma_i = 0$ for all brands, and then computing the AIC and AIC_C values. For this model, AIC = 1849.8 and AIC_C = 1992.48 (see Table 2). The model comparison shows that price-effects model without interaction effects is not as good as the price-effects model (9). This finding, together with the earlier result, indicates that the interaction effects, more so than price effects, play a significant role in this dynamic competitive market for detergents.

3.4.4 *Diminishing Returns*

As noted earlier, the basic Lanchester model (1) incorporates the notion of diminishing returns. We incorporate additional diminishing-returns via concave transformations. For example, $u = \sqrt{x_1}$ and $v = \sqrt{x_2}$, where (x_1, x_2) denote the advertising and promotion variables, respectively. We re-estimate the dynamic model with interactions using these transformations, thus obtaining AIC = 1775.34 and AIC_C = 1876.25. Based on these AIC and AIC_C values, we retain the proposed dynamic model with interactions over this alternative one.

We further test the sensitivity to the specification for diminishing returns by considering logarithmic transformations, $u = \text{Ln}(1 + x_1)$ and $v = \text{Ln}(1 + x_2)$. (We use unity in the arguments so that the log function remains well behaved when advertising or promotion is absent in some week.) We re-estimate the dynamic model with log transformations. Comparing the values of information criteria (see Table 2), we retain the proposed dynamic model with interactions.

3.4.5 *Increasing Returns*

To check the possibility of increasing-returns, we specify the convex transformations $u = x_1^2$ and $v = x_2^2$. We compute the information criteria AIC and AIC_C (see Table 2) and, upon models comparison, we retain the dynamic model with interactions.

In sum, we find that the proposed dynamic model with interactions is robust to the above perturbations of model specifications. More importantly, we illustrate that the continuous-discrete estimation approach can be applied to calibrate a broad class of dynamic models for markets with multiple brands (i.e., oligopolistic competition), using multiple marketing activities (e.g., price, promotion, and advertising) under various market response conditions (e.g., diminishing returns, increasing returns, and interaction effects). After calibrating and selecting the appropriate dynamic market response model, managers need to plan their marketing-mix strategy with strategic foresight. To this end, we next conduct dynamic game-theoretic analysis, and provide an implementable computational algorithm.

4. EQUILIBRIUM ANALYSIS

In this section, we formulate the differential-game, derive the equilibrium strategies, construct a computational algorithm, and present the normative results.

4.1 Differential Game Formulation

Each brand i , $i = 1, \dots, N$, develops its advertising plan $u_i(t)$ and promotion plan $v_i(t)$ to maximize the performance index (Erickson 1991, p. 18)

$$(10) \quad J(u_i, v_i) = \int_{t=0}^T \underbrace{e^{-\rho t}}_{\text{discounting}} \underbrace{[(p_i(t) - v_i(t))m_i(t) - c(u_i(t))]}_{\substack{\text{net value} \\ \text{discounted net value}}} dt,$$

which quantifies the performance of the strategies $(u_i(t), v_i(t))$. In equation (10), ρ denotes the discount rate, p_i is the price of brand i ³, v_i is the size of the deal, m_i is the market share, u_i is the advertising effort, and $c(u)$ is the cost of expending the effort u .

A strategy pair (u, v) is a time-path over the planning horizon T . A brand i 's strategy influences its own brand share; see the equations (3) and (5). Moreover, it influences the market shares of all other brands because of the term F in each brand's market share dynamics in equation (3). Hence, each brand i should incorporate strategic foresight in planning a good marketing strategy. That is, it takes into account not only the effect of its own strategy (u_i, v_i) on its market share, but also the effects of all other brands' strategies (U_{-i}, V_{-i}) , where U_{-i} and V_{-i} represent the set of advertising and promotion strategies, u_j and v_j , respectively, for all other brands $j, j \neq i$.

The formal problem facing each brand i is to find the equilibrium advertising and promotion strategies (u_i^*, v_i^*) , which maximize the performance index J_i and incorporate strategic foresight. We solve this problem by constructing the Hamiltonian and deriving the Nash equilibrium strategies.

4.2 Hamiltonian and Equilibrium Strategies

4.2.1 Hamiltonian

³ Firms incur several expenses such as packaging costs, pass-through adjusted trade promotion costs, distribution and logistics, and other allocated overheads per unit, all of which can be included in the unit cost term $c_i(t)$. In practice, managers should use the net price $\tilde{p}_i(t) = p_i(t) - c_i(t)$ to account for such costs of goods sold. For purposes of constructing methods to solve differential games, we assume them to be zero with no loss of generality and consistent with extant models of promotion (see Gerstner and Hess 1991, p. 873).

We recall that, in static optimization problems, the objective function is augmented by adjoining Lagrange multipliers to incorporate static constraints. Similarly, to solve the dynamic optimization problem, we construct the Hamiltonian function, in which we adjoin the dynamic constraint (3) to the objective function (10). Thus, for each brand i , we define the Hamiltonian as follows:

$$(11) \quad H^i(u_i, v_i) = (p_i - v_i)m_i - c(u_i) + \lambda_i(f_i - Fm_i), \quad i = 1, \dots, N,$$

where $\lambda_i(t)$ is the co-state variable for brand i , and f_i is brand i 's marketing force, where

$$f_i = \alpha_i u_i + \beta_i v_i + \gamma_i u_i v_i \text{ (from equation 7) and the total market force } F = f_i + \sum_{\substack{j=1 \\ j \neq i}}^N f_j. \text{ In the}$$

following analysis, we assume the cost function $c(u) = bu + u^2 / 2$ (see, e.g., Fruchter and Kalish 1998), although the proposed computational algorithm accommodates other functional forms (see the subsection 4.3).

4.2.2 Equilibrium Strategies

4.2.2.1 Optimal Advertising

By differentiating the Hamiltonian (11) with respect to u_i , and setting $\partial H^i / \partial u_i = 0$, we obtain

$$(12) \quad u_i^* = \lambda_i(\alpha_i + \gamma_i v_i)(1 - m_i) - b, \quad i = 1, \dots, N.$$

Equation (12) specifies the optimal advertising strategy for each brand i , and shows that it depends on one's own market share (m_i), ad effectiveness (α_i), and the cost parameter b . In addition, it depends on all other brands' strategies via the co-state variable $\lambda_i(t)$, whose dynamics will be specified later (see the equation 17).

We observe from equation (12) that, in the absence of interaction effects (i.e., when $\gamma_i = 0$), advertising plans do not directly depend on the promotion plans. In contrast, in the presence of interaction effects (i.e., when $\gamma_i \neq 0$), optimal advertising $u_i^*(t)$ depends directly on the promotion plan $v_i(t)$. Hence, the importance of interactions — they qualitatively change the planning of the marketing-mix by requiring managers to incorporate inter-activity tradeoffs.

The substantive implication is that managers need to take into account the inter-dependence between advertising and promotion in order to plan their strategies optimally. We emphasize that this inter-dependence would not arise in models that ignore the interaction effects. Moreover, this result is interesting in its own right, because it is not driven by a budget constraint on the two activities (which would make this dependence obvious and thus uninteresting).

4.2.2.2 Optimal Promotion

By differentiating the Hamiltonian (11) with respect to v_i , we obtain

$$(13) \quad \partial H^i / \partial v_i = -m_i + \lambda_i(1 - m_i)(\beta_i + \gamma_i u_i),$$

but we cannot set it to zero to solve for the optimal promotion plan $v^*(t)$. This is because the equation (11) is linear in v , and so the right-hand side of (13) does not depend on v . Hence, we define the switching function,

$$(14) \quad D_i = -m_i + \lambda_i(1 - m_i)(\beta_i + \gamma_i u_i),$$

which yields the optimal bang-bang policy,

$$(15) \quad v_i = \begin{cases} \bar{v}_i & D_i > 0 \\ 0 & D_i < 0 \end{cases}, \quad i = 1, \dots, N$$

where \bar{v}_i denotes the maximum deal amount for brand i at time t . Thus, the optimal promotion plan is either “on” (i.e., offer a price discount) or “off” in any given week, depending on the sign of the

switch D in equation (14). (Note that $D_i \in \mathfrak{R}^1$, and so $D_i = 0$ in continuous-time models is a measure-zero event, which substantively means that a brand i 's strategy to “promote” and “not promote” at the same time is not sustainable.)

A promotion plan given by (15) can be compactly expressed as $v_i(t) = \bar{v}_i I(D_i > 0)$, where $I(\cdot)$ denotes an indicator variable that takes the value 1 or 0 depending on whether the switch D is positive or not. This switch depends on the co-state variable $\lambda_i(t)$, which involves the strategies of all other brands (see equation (17)). Consequently, each brand forms an expectation of whether the other brands would promote or not. By taking expectations, we find that brand i 's policy is given by

$$\begin{aligned}
 v_i^* &= \bar{v}_i E[I(D_i > 0)] \\
 &= \bar{v}_i \Pr(-m_i + \lambda_i(1 - m_i)(\beta + \gamma u_i) > 0) \\
 (16) \quad &= \bar{v}_i \Pr(m_i < \frac{\lambda_i(\beta + \gamma u_i)}{1 + \lambda_i(\beta + \gamma u_i)}) \\
 &= \frac{\bar{v}_i \lambda_i(\beta + \gamma u_i)}{1 + \lambda_i(\beta + \gamma u_i)},
 \end{aligned}$$

where the last equality follows from the observation that the brand share m_i has a support on the unit interval $(0, 1)$, which is akin to the standard uniform random variable U for which $\Pr(U < c) = c$.

Finally, to obtain the equilibrium strategies, we iteratively determine the fixed-point of the equations (12) and (16). The resulting equilibrium pair (u^*, v^*) depends on the co-state variables $\lambda(t)$, which are key to capturing the notion of strategic foresight.

4.2.2.3 Co-state Dynamics

A fundamental principle of game theory is strategic foresight, requiring managers to look forward (i.e., anticipate competitive moves), and reason backwards (i.e., deduce the best strategy via backward induction). To incorporate strategic foresight in the equilibrium strategies (u^*, v^*) , we specify the dynamics for co-state variables, which are given by

$$(17) \quad \begin{aligned} \dot{\lambda}_i &= \frac{d\lambda_i}{dt} = \rho\lambda_i - \frac{\partial H^i}{\partial m_i} \\ &= (\rho + F)\lambda_i - (p_i - v_i), \end{aligned}$$

where $i = 1, \dots, N$. The last equality arises from differentiation of (11) with respect to m_i . We note that (17) represents a set of differential equations that are neither linear nor independent of each other (i.e., their simple appearance is deceptive). We get nonlinearity because $\dot{\lambda}_i$ depends on not only λ_i , but also (F, f_i, v_i) which, in turn, are functions of λ_i via the equations (5), (12) and (16). In addition, they are coupled because F includes a competitor's marketing force f_j , which are functions of the competitor's decisions (u_j, v_j) which, in turn, depend on the corresponding co-state variable λ_j . Hence, we must solve all the differential equations in (17) simultaneously.

We interpret the co-state variable λ as follows: it measures the marginal change in the performance index (10) due to a marginal change in the rate of growth of brand share.

Essentially, it systematically captures the profit impact due to the changes in brand share, which result from the brand's own and its competitors' advertising and promotion strategies.

Consequently, each brand's co-state variable depends directly on not only its own marketing-mix

plans, but also all other brands' strategies via the total market force $F = f_i + \sum_{\substack{j=1 \\ j \neq i}}^N f_j$. Thus, the

construct of "co-state dynamics" permits each brand to anticipate formally (i.e., in the game-

theoretic sense) the profit impact of competitive decisions of all other brands over time and

across marketing activities. For example, in equation (17), we account for not only advertising

and promotion decisions of all brands (via F), but also margin reductions (via $p_i - v_i$).

In equation (18), we have $(N-1)$ state equations (since all shares sum to unity), and N co-state equations. That is, we have a system of $(2N-1)$ coupled, nonlinear, differential equations. For example, with five brands, we obtain a 9-dimensional vector differential equation.

Because equation (18) satisfies boundary conditions at two different time-points — the initial and the terminal points of the planning horizon — we refer to it as the two-point boundary value problem (TPBVP). Specifically, the brand share dynamics satisfies the initial values

$$(19) \quad (m_1(0), \dots, m_i(0), \dots, m_{N-1}(0))',$$

and the co-state dynamics satisfy the terminal values

$$(20) \quad (\lambda_1(T+1), \dots, \lambda_i(T+1), \dots, \lambda_N(T+1))'.$$

The above problem of planning marketing-mix for N competing brands in the presence of interactions does not permit a closed-form solution. Hence, we next present a computational algorithm to solve it numerically.

4.3.2 Computational Algorithm

The extant algorithms to solve marketing TPBVPs (e.g., Deal 1979, Thompson and Teng 1984, Erickson 1991) ignore the coupling between and within the state and co-state equations.

However, this coupling manifests itself naturally due to the presence of the total market

force $F = f_i + \sum_{\substack{j=1 \\ j \neq i}}^N f_j$ in every state and co-state equations in (18), and all the f_i are functions of $(m_i,$

$\lambda_i)$ via the equations (12) and (16). Hence, we present an algorithm that accounts for such inter-equation coupling. More generally, the proposed algorithm accommodates various formulations:

- different state dynamics, instead of the extended Lanchester model in (3);
- different marketing forces, instead of the equation (5);
- other marketing objectives, instead of the performance index (10);

- other marketing phenomena (e.g., channel, reference price, cost functions); and
- alternative solution concepts (e.g., closed-loop policy).

To achieve this generality, we abstract the equation (18) to a more general object

$$(21) \quad \dot{y} = g(y; \Theta).$$

It is important to recognize that, despite the simple appearance of (21), many differential-games in marketing (see Erickson 1995) can be eventually cast in this form — only the shape of the link function $g(\cdot)$ and the set of input arguments (y, Θ) would change to accommodate different state dynamics, marketing forces, objectives, phenomena, and solution concepts. For example, to plan the marketing-mix strategies for 5 brands, the link function $g(\cdot)$ is specified by (18), the vector $y = (m_1, m_2, m_3, m_4, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$, and the set Θ contains the initial conditions (19), the terminal conditions (20), the parameter estimates (see Panel A of Table 1), and the known price history. Incorporating the recent advances in solving TPBVP (see Ascher, Matheij, and Russell 1995), we construct a computational algorithm that consists of six steps, which are presented in Table 3.

[Insert Table 3 here]

4.3.3 *Properties of the algorithm*

In step 1 of Table 3, we chose $M = 84$ grid points so that the optimal solutions are available for each week of the observed data. In principle, the number of grid points M can be less or more than the sample size T (i.e., user can specify coarser or finer grid). When the grid is fine, the starting values (in step 4) for the intermediate points can be obtained by interpolation. To illustrate the empirical application, we use discount rate of 13% per annum ($\rho = 0.0025$ per week) and the cost parameter $b = -1$.

In step 2, the number of finite-difference equations is $p = (2N-1) \times M$, where N is the number of brands. Consequently, the vector function $\mathbf{G}(\mathbf{x})$ is a mapping $G: \mathfrak{R}^p \rightarrow \mathfrak{R}^p$. Step 2

replaces the problem of solving coupled nonlinear differential equations in (21) with the problem of finding roots of the system of algebraic equations $\mathbf{G}(\mathbf{x}) = \mathbf{0}$. Thus, by increasing the dimensionality from $(2N-1)$ to $(2N-1) \times M$, we trade-off harder-to-solve differential equations with easier-to-solve algebraic equations.

In step 3, we apply the standard Newton-Raphson (NR) method to find roots. The root-finding problem is easier to solve because we know the starting values (see step 4), and the Jacobian of $\mathbf{G}(\mathbf{x})$ is a block diagonal matrix, which inverts easily due to its sparse structure. Consequently, we attain convergence in 3-4 iterations, each iteration taking less than 20 seconds on a desktop computer. In other words, managers can obtain annual marketing-mix plans in a few minutes.

Step (6b) deserves special mention. Consider the annual planning horizon of 52 weeks in a market with five brands. A brand can choose to promote or not promote in the first week. Given this decision, the brand can promote or not promote in the second week; and so on. Proceeding in this manner, each brand manager faces $2^{52} \approx (2^{10})^5 > 10^{15}$ possibilities, which represents thousands of trillions of ways to plan promotion calendars. The real task is even more challenging: it requires each brand to consider as many possibilities for the other brands, leading to $10^{15 \times 5}$ strategies for discovering a few good promotion plans. Even modern computers cannot enumerate all these counter-factual scenarios, let alone human decision-makers (i.e., media planners in ad agencies). Hence, it is remarkable that the proposed algorithm finds the equilibrium promotion plan so efficiently. The power of the algorithm stems from the notion of switching function, which we derive in the equation (15) and use it in step (6b). Without its contribution, the task of determining equilibrium promotion plans over realistic planning horizons would remain unsolvable.

Finally, and most importantly, this algorithm is not restricted to the extended Lanchester model (3). Indeed, managers can apply it to other market response models that capture the essence of the marketing environment specific to their companies. This generality stems from the fact that this algorithm solves the TPBVP implied by equation (21), which subsumes a variety of dynamic models for competitive markets.

4.4 Normative Results

4.4.1 Actual vs. Optimal Advertising

Managers can compare their actual advertising and promotion decisions relative to other brands. However, this relative benchmarking does not yield diagnostic information on how close or far they are from being optimal. In contrast, Table 4 presents information on both the actual and optimal decisions. Specifically, for each brand, Panel A presents the actual versus optimal advertising effort, and Panel B presents that information for the promotion effort. Panel C summarizes the resulting impact on the performance criterion (10). Consistent with the empirical analyses, we report the standardized effort levels for advertising and promotion (e.g., actual cents-off divided by its standard deviation).

By comparing the columns in Panel A, we observe the regularity that actual advertising effort is below the optimal level. Across the five detergents brands, the average under-advertising is 25.26%, and it varies from 10.07% to 43.75%. Our results are in contrast to the belief that marketers overspend on advertising (e.g., Aaker and Carman 1982), which is based on studies conducted in periods during which escalation in promotion spending was not severe. Finally, we notice that the larger brands (e.g., Tide and Wisk) under-advertise more than the smaller ones (e.g., Era and Solo).

[Insert Table 4 here]

4.4.2 Actual vs. Optimal Promotion

Unlike advertising, where all brands uniformly under-advertise, we observe a mixed promotional response: some brands under-promote, while others over-promote. Panel B reveals an interesting finding that Tide and Wisk, who under-advertise substantially, tend to promote their brands excessively. This result is interesting because the normative analysis considers advertising and promotion decisions jointly, and yet finds that large brands allocate fewer resources than they should to advertising, relative to promotion. Thus, our analysis provides direct supporting evidence for Leeflang and Wittink's (2001, p. 120) remark on the escalation in promotion expenditures due to managers' lack of strategic foresight in planning the marketing-mix. On the other hand, small brands such as Era and Solo under-promote and under-advertise, perhaps due to the lack of resources.

[Insert Figure 1 here]

4.4.3 *Switching Functions*

The switching function enables managers to decide whether to promote in any given week: when the value of the switch is positive, promotion is on; when negative, promotion is off. Panel A of Figure 1 shows the optimal switching functions in the presence of interactions. For Wisk, Tide and Bold, the switch $D(t)$ is always negative, and so managers should not promote these brands. Rather, they should maintain a constant — but high — price every week of the planning horizon. For Solo, by contrast, the switch $D(t)$ is always positive, suggesting that managers should maintain a constant — but low — price every week (i.e., the every-day-low-price (EDLP) strategy).

For Era, however, the switch $D(t)$ changes sign several times during the planning horizon. Hence, Era should apply the pulsing strategy, i.e., promote for a few weeks, then stop promotion for a few weeks, then promote again (see the dotted line in Panel A). This finding of optimality

of pulsing strategy is new. Moreover, it is interesting because previous advertising studies (see, e.g., Mahajan and Muller 1986, Feinberg 2001) show that pulsing is *not* optimal for a broad class of dynamic response models.

Panel B of Figure 1 shows the optimal switching functions in the absence of interactions. As in Panel A, Tide and Wisk should not promote at all, thus maintaining the every-day-high-price strategy. Similarly, Solo should follow its EDLP strategy, while Era continues with its pulsing strategy. However, for Bold, the switching function shifts upwards (see the bold line), and it now becomes optimal to change the strategy from every-day-high-price to pulsing strategy. (The optimal start and stop times are determined by noting when the switching function crosses the timeline in Panel B of Figure 1.) Thus, by ignoring the interaction effects, some brands (e.g., Bold) would run price promotion rather than maintain high price, thereby decreasing profit (see Panel C in Table 4 for performance of these strategies).

5. DISCUSSIONS

This section elaborates important issues, and suggests topics for further research.⁴

5.1 Testing Exogeneity

We estimated the Lanchester model by assuming that advertising and promotion are exogenous variables for each of the five brands. Alternatively, managers' spending decisions may depend (endogenously) on previous market shares. If the latter behavior prevails in the marketplace, while we ignore it in the model estimation, then the estimated model may yield incorrect estimates (e.g., Bass 1969, Villas-Boas and Winer 1999). Hence, we test the hypotheses whether or not advertising and promotion are exogenous.

⁴ We thank both the AE and reviewers for their valuable suggestions.

We apply the concepts of weak- and super-exogeneity developed by Engle, Hendry and Richard (1983). To clarify these concepts, suppose we factorize the joint density of brand share and advertising $f(Y, X)$ into conditional density of brand share given advertising $g(Y|X)$ and marginal density of advertising $h(X)$. Then, weak-exogeneity of advertising means that a precise specification of the marginal density h is not necessary and model estimation via the conditional density g results in no loss of information. In addition, advertising is super-exogenous if model parameters remain constant over time. (These notions apply to brand promotion also.)

Appendix A furnishes the details on testing for exogeneity, and we note here the main findings. Both advertising and promotion are weakly exogenous for all the five brands, and hence the empirical results are valid in the sense of consistent and efficient estimation. In technical words, parameter estimates are asymptotically unbiased and possess minimum variance. In addition, all the five brands satisfy super-exogeneity requirements, and thus the retained model is adequate for policy simulation.

5.2 *Optimal Competitive Responsiveness*

Managers can gain further insights into competitive responsiveness by conducting comparative static analyses as follows. For example, if Tide's advertising becomes more effective, should Wisk respond with increased advertising, or with increased promotion? Should Solo's response be similar to, or different from, Tide's? To this end, managers should first characterize optimal advertising and promotion policies for each brand (via the proposed computational algorithm), and compute the total effort on advertising and promotion over the planning horizon. Then, they should increase any specific parameter (say, ad effectiveness of Wisk), keeping all other parameters constant, and find the new optimal policies for all brands. They can thus assess whether

the new policies indicate an increase or decrease in total advertising and promotion efforts, and evaluate the resulting impact on a brand's performance index.

Conducting comparative statics, we present results in Table 5 for the interaction effects γ_i ($i = 1, \dots, 5$). Based on Table 5, brand managers not only understand their own optimal actions, but also anticipate other brands' optimal competitive responsiveness. Table 5 reveals that competitive responsiveness is asymmetric in this detergents market. For example, if negative interaction effect of Tide increases, its own allocation behavior and other brands' best response (see Tide's column in Table 5) are different from those due to an increase in Wisk's (say) negative interaction effect (see the column for Wisk). Managers can also understand the impact of counter-factual scenarios on brand performance. For example, how would a marginal increase in Solo's negative interaction affect own and other brands' performance? Based on this market dataset, Solo's profit would decline, benefiting the other brands (see Table 5, column Solo). Finally, besides the interaction effects, managers can learn competitive responsiveness to changes in ad effectiveness or promotion effectiveness of any brand via comparative statics. (We do not present or elaborate these results here because of space constraints.)

[Insert Table 5 here]

5.3 *Closed-loop Strategies*

We apply the open-loop solution concept to determine optimal plans as functions of time so that our analyses comport with the prevailing institutional practice. Specifically, ad agencies buy media time for specific temporal slots (e.g., day parts of television shows, prime time programs, summer time re-runs, fall shows, Super Bowl ads), and execute media contracts on behalf of their clients a year in advance, a practice known in industry as "up-front market" (e.g., Belch and Belch 2001, p. 364-365). Similarly, promotion calendars roll out over specific weeks of the year, allowing

sales force to communicate the planned activities with channel members (Silva-Risso, Bucklin, and Morrison 1999).

However, if one seeks closed-loop strategies, the computational algorithm in Table 3 can generate them; that is, this algorithm is **not** restricted to open-loop strategies only. Before we substantiate this claim, we note here two substantive implications for managers in pursuing closed-loop strategies. First, media institutions such as network stations (e.g., NBC) need to change the media rate cards from time-based to share-based tariffs. In other words, NBC should agree to sign media contracts with advertisers such as Procter and Gamble to sell more or less airtime depending on changes in a brand share relative to competing brands. In addition, one has to design enforceable contracts for eventualities where a brand (e.g., Wisk) could cancel its airtime, not because its own share changed, but because other competing brands gained or lost share points (e.g., Tide lost share to Era). This flexibility seems far-fetched in current media buying-selling environment, which demands strong commitments from advertisers. Secondly, brand managers need to institute a measurement system on a systematic basis to provide week-by-week information on sales, advertising and promotion for all competing brands. Presently, brand managers know their own sales on a weekly basis, but not those for other brands — the estimates of competitor’s sales and advertising trickle slowly over the years on ad hoc basis from market research. To express it differently, closed-loop strategy is information intensive (Dockner, Jorgensen, Van Long, and Sorger 2000, p. 29): each brand’s value function depends on all brands’ shares, $V^i(\mathbf{m})$ where $\mathbf{m} = (m_1, \dots, m_N)'$.

Now, to illustrate how to obtain closed-loop strategy via the algorithm in Table 3, we let V^i denote the value function for brand i , and apply Dockner et al. (2000, p. 95) to get

$$(22) \quad \rho V^i = (p_i - v_i)m_i - c(u_i) + \dot{V}^i(f_i - Fm_i), \quad i = 1, \dots, N$$

where $\dot{V}^i = \partial V^i / \partial m_i$, the controls (u, v) and the marketing forces (f, F) are evaluated at their optimal values. Note that Hamiltonian in equation (11) is analogous to equation (22) via the identification of \dot{V}^i with the co-state variable λ_i . The main difference, however, is that each of the N equations in (22) relates the level V^i to the partial derivative $\partial V^i / \partial m_i$, thus resulting in *partial* differential equations (in contrast to ordinary differential equations in (18)). Furthermore, each equation in (22) is nonlinear because the marketing force f_i depends on the optimal controls (u_i, v_i) , which are themselves functions of \dot{V}^i via the equations (12) and (16). In addition, all equations in (22) are coupled to each other because the total marketing force F depends on optimal controls (u_j, v_j) of all brands. We denote this system of nonlinear and coupled partial differential equations (PDEs) in the general notation:

$$(23) \quad \tilde{g}(\dot{y}, y, \Theta) = 0,$$

where $\tilde{g}(\cdot)$ is a vector-valued implicit function of partial derivatives of variables (\dot{y}) , levels of variables (y) , and parameters (Θ) .

Applying the computational algorithm in Table 3, we solve the nonlinear and coupled PDEs in equations (22) or (23) simultaneously (see Smith 2003). To this end, as before, we generate a set of grid points, express derivatives by their forward differences, obtain algebraic equations based on (22) or (23), and then find roots via the Newton-Raphson method.

To characterize the closed-loop strategies, consider first the pairs of brands Wisk and Tide, Tide and Era, and so on. For any specific pair, say Wisk and Era, visualize competition for market share on the unit interval $(0, 1)$ and choose $M = 11$ grid points (i.e., $k = 1, 2, \dots, 11$) for Wisk's market share from zero to unity with a step size $h = 0.1$. For each pair of brands ($i = \text{Wisk, Era}$), using equation (22), relate algebraically the value V_k^i with its forward-difference estimate for \dot{V}_k^i on

each grid point $k = 1, \dots, 11$. Together with the two boundary values $V_1^{\text{Wisk}} = 0$ and $V_M^{\text{Era}} = 0$, solve the resulting 22 ($= 11 \times 2$) algebraic equations to estimate V_k^i for each grid point k ($k = 1, \dots, 11$), which corresponds to market shares $0(0.1)1$. Knowing V^i as a function of market shares, we can determine the optimal closed-loop advertising strategy from equation (12) and closed-loop promotion strategy from equations (14)-(16). Similarly, estimate the value functions V_k^i for the other pairs (Wisk and Tide, Tide and Era), and determine the closed-loop strategies as functions of market shares. These closed-loop strategies hold for duopolistic markets consisting of pairs of brands, and serve as boundary functions (akin to boundary values) for the triopoly market.

Consider next the triopoly, Wisk, Tide and Era. Here we visualize competition for market share on the unit square $(0, 1)^2$ with horizontal and vertical axes representing Wisk and Tide shares, respectively. Because shares sum to unity, competition prevails over the right triangle shown in Panel A of Figure 2. We observe that the Wisk-Era duopoly plays out on the horizontal axis (where Tide share is zero), the Tide-Era duopoly unfolds on the vertical axis (where Wisk share is zero), and the Wisk-Tide duopoly exists on the hypotenuse (where Era's share is zero). As before, choose 11 discrete points for Wisk market share from zero to unity with a step size $h = 0.1$, and similarly 11 points for Tide's market share. This discretization results in square meshes (of side 0.1) inside the right triangle. As shown in Panel A of Figure 2, attach numbers 1 through 30 to the boundary points of the right triangle, and numbers 1 through 36 to its interior points. Applying the algorithm in Table 3, formulate first the algebraic equations for V_k^i , $k = 1, \dots, 36$, where $i = \text{Wisk, Tide, and Era}$. We then solve the resulting 108 ($= 36 \times 3$) equations to estimate V_k^i for each grid point k ($k = 1, \dots, 36$) and the three competing brands.

[Insert Figure 2 here]

Panels B, C and D in Figure 2 display the results graphically for the Wisk brand. Panel B displays the contour plot for the estimated value function $V^{\text{Wisk}}(m_{\text{Wisk}}, m_{\text{Tide}}, m_{\text{Era}})$. It is important to recognize that the area below the diagonal accounts for the presence of Era, whose share is $m_{\text{Era}} = 1 - m_{\text{Wisk}} - m_{\text{Tide}}$. Knowing V^i as functions of brand shares, we can determine the closed-loop advertising strategy from equation (12) and closed-loop promotion strategy from equations (14)-(16). Panel C presents the contour plot for Wisk's closed-loop advertising strategy. It shows whether Wisk should advertise at low, medium or high levels for every combination of brand shares. Similarly, Panel D shows its closed-loop promotion strategy, i.e., whether Wisk should promote or not for every combination of brand shares. Although we estimated the value functions V_k^i for Tide and Era as well as their (closed-loop) advertising and promotion strategies, we do not present them because of space constraints. Proceeding in this manner, we could obtain closed-loop strategies for other brands via the computational algorithm, which generalizes to higher dimensions, even though we cannot visualize graphically the solutions for $N > 3$ brands.

In sum, the computational algorithm is versatile: it finds both the open- and closed-loop strategies. This is because the mathematics underlying this algorithm treats each grid point as a state of the dynamic system, and “the mathematics does not care” (Little 1986, p. 107) whether grid points are “weeks” (in open-loop strategies) or “shares” (in closed-loop strategies). Hence, via this algorithm, managers can obtain either open-loop or closed-loop strategies, a choice they should make based on the institutional aspects of their operating environment (see Dockner et al., p. 29).

5.4 *Further research: the role of dynamic retailers?*

While the extant literature has investigated important channel issues, it tends to ignore the effects of temporal dynamics, marketing-mix interactions, or strategic foresight. For example, Neslin, Powell, and Stone (1995) incorporate a rich model structure in terms of temporal

dynamics, marketing-mix variables, and retailer pass-through behavior, but they do not incorporate strategic foresight of manufacturers and retailers (see p. 765). On the other hand, Lee and Staelin (1997) propose a game-theoretic structure to understand strategic foresight between two manufacturers and two retailers, but they abstract away the roles of temporal dynamics, multiple marketing-mix variables, interaction effects, and multiple brands. Similarly, Kopalle, Mela and Marsh (1999) investigate the Stackelberg game between the manufacturer and retailer over time. However, the focal manufacturer conjectures the prices of competing brands via price reaction functions, and so their price-trajectories do not incorporate strategic foresight of other brands. In addition, Kopalle et al. (1999) focus on price as the sole decision variable, thus ignoring both main and interaction effects of other marketing-mix variables (e.g., advertising). Hence, future research can augment the scope of these models by formulating differential games and incorporating marketing-mix interactions. In other words, it is important to incorporate *both* the rich model structure (e.g., Neslin, Powell and Stone 1995) and the game-theoretic structure (e.g., Lee and Staelin 1999). Such manufacturer-retailer differential games will result in nonlinear boundary value problems, which we can express as special cases of equations (21) or (23) and hence solve them via the computational algorithm given in Table 3. We thus emphasize that the estimation and computational approaches developed here pave the way to obtain systematically optimal strategies not only for such manufacturer-retailer differential games, but also for many other dynamic marketing problems (e.g., online auctions).

6. CONCLUDING REMARKS

This paper develops a methodology for planning marketing-mix in dynamic competitive markets, incorporating strategic foresight and interaction effects. The notion of strategic foresight requires each brand manager to anticipate likely decisions of competing brands and deduce its own optimal strategy in response to the best decisions of all other brands. The marketing-mix concept requires managers to recognize that multiple marketing activities (e.g., advertising, promotion) not only affect market shares directly, but also amplify or attenuate the effectiveness of marketing activities indirectly. The incorporation of strategic foresight and interaction effects in a planning problem raises challenging issues in both empirical and equilibrium analyses. Consequently, without the aid of decision-making tools, managers find optimal strategic reasoning an unnatural task (Urbany and Montgomery 1998). Hence, we designed an appropriate Kalman filter to estimate the multivariate continuous-time dynamic models using discrete-time observed data, and constructed a computational algorithm to solve a general nonlinear two-point boundary value problem for determining optimal marketing-mix plans. The generality of this framework empowers managers by providing a technology to tackle realistic marketing problems in dynamic oligopolistic markets.

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TABLE 1. CONTINUOUS-DISCRETE KALMAN FILTER ESTIMATES

PANEL A. *Dynamic Model with Interaction Effects*

<i>Parameters</i>	<i>Tide</i>	<i>Wisk</i>	<i>Era</i>	<i>Solo</i>	<i>Bold</i>
Ad Effectiveness, α_i	0.3324 (5.07)	0.1667 (2.67)	0.0734 (3.50)	0.0526 (3.76)	0.0868 (1.35)
Promotion Effectiveness, β_i	0.1169 (4.42)	0.0369 (3.03)	0.0508 (3.29)	0.0520 (4.93)	0.0512 (1.77)
Ad-Promotion Interaction, γ_i	-0.0954 (-3.94)	-0.0164 (-0.53)	0.0032 (0.30)	-0.0127 (-1.34)	-0.0203 (-0.57)
Transition Noise, $\exp(b_i)$, b_i	-4.7041 (-24.82)	-5.2112 (-21.01)	-5.4118 (-19.46)	-6.3022 (-16.53)	--
Observation Noise, σ	2.9584 (2.77)				
Maximized Log-likelihood, LL*	-852.55				
AIC	1745.01				
Bias-corrected AIC _C	1845.99				

PANEL B. *Dynamic Model without Interaction Effects*

<i>Parameters</i>	<i>Tide</i>	<i>Wisk</i>	<i>Era</i>	<i>Solo</i>	<i>Bold</i>
Ad Effectiveness, α_i	0.1261 (4.41)	0.1178 (5.90)	0.0605 (4.29)	0.0305 (3.35)	0.0438 (1.38)
Promotion Effectiveness, β_i	0.0709 (3.64)	0.0243 (2.41)	0.0442 (3.54)	0.0383 (5.31)	0.0435 (1.66)
Transition Noise, $\exp(b_i)$, b_i	-4.5997 (-23.57)	-5.2872 (-20.99)	-5.5444 (-18.31)	-6.6333 (-15.57)	--
Observation Noise, σ	3.7107 (4.88)				
Maximized Log-likelihood, LL*	-862.853				
AIC	1755.71				
Bias-corrected AIC _C	1849.83				

NOTE: The numbers in parentheses are t-values.

TABLE 2. INFORMATION CRITERIA FOR VARIOUS MODELS OF DYNAMIC OLIGOPOLY

<i>Models</i>	<i>AIC</i>	<i>AIC_C</i>
Dynamic Model with Interaction Effects	1745.01	1845.99
Dynamic Model without Interaction Effects	1755.71	1849.83
Price Effects with Interaction Effects	1816.01	1984.01
Price Effects without Interaction Effects	1849.80	1992.48
Diminishing Returns (squared-root)	1775.34	1876.25
Diminishing Returns (logarithmic)	1809.14	1910.04
Increasing Returns (quadratic)	1798.14	1899.05

TABLE 3. COMPUTATIONAL ALGORITHM

Step 1	Boundary conditions
a	Choose M grid points on the timeline, $k = 1, 2, \dots, M$.
b	Specify the boundary conditions at the initial and terminal points: $k = 0$ and $k = M + 1$. Specifically, for each brand i , the initial market share is $m_i(0) = \bar{m}_i,$ where the bar notation denotes an average over time. Similarly, the terminal co-state value for each brand is $\lambda_i(T + 1) = \frac{\bar{p}_i - \bar{v}_i}{\rho + \bar{F}},$ which represents the steady-state of the equation (17) evaluated at the average market conditions.
Step 2	Finite difference equations
a	Use equation (21) to create the difference equation: $E_k \equiv y_k - y_{k-1} - h g(w_k; p_k, \theta),$ where $h = t_k - t_{k-1}$, and $w_k = (y_k + y_{k-1})/2$. E_k denotes a set of nine equations (i.e., the dimension of vector y). Repeat this process for $k = 2, 3, \dots, M$ to obtain E_2, E_3, \dots, E_M . Thus, this step yields a total of $9 \times (M-1)$ equations. Also, note that we denote Θ in (21) by (p_k, θ) to indicate explicitly the presence of time-varying prices for all brands.
b	Use the initial conditions from step (1b) to obtain 4 more equations: $E_1 \equiv z' [y_1 - y_0 - g(w_1; p_1, \theta)],$ where $z = (1,1,1,1,0,0,0,0,0)'$ selects the first four equations.
c	Use the terminal conditions from step (1b) to obtain 5 more equations: $E_{M+1} \equiv z' [y_{M+1} - y_M - g(w_M; p_M, \theta)],$ where $z = (0,0,0,0,1,1,1,1,1)'$ selects the last five equations.
d	Create a vector \mathbf{G} by stacking all the equations $E_1, E_2, \dots, E_M, E_{M+1}$ one below another. Let the vector $\mathbf{x} = \text{vec}(y_1, y_2, \dots, y_M)$. Thus, steps (2a, 2b, 2c) lead to the vector function $\mathbf{G}(\mathbf{x})$, which consists of $9 \times M$ equations in $9 \times M$ variables.
Step 3	Nonlinear root-finding
	Solve the system of nonlinear equations: $\mathbf{G}(\mathbf{x}) = \mathbf{0}$. To solve it, apply root-finding procedure such as the Newton-Raphson method using the following starting values.
Step 4	Starting values
a	Let the starting values for the state variables be the actual brand shares m_{ik} for

-
- b each brand $i, i = 1, \dots, 5$, and for each $k = 1, \dots, T$.
 Compute the starting values for the co-state variables using equation (17).

Specifically, use the terminal values in step 1b, and apply the backward recursion:

$$\lambda_i(k) = e^{-(\rho+F_k)} \lambda_i(k+1) + \frac{1 - e^{-(\rho+F_k)}}{\rho + F_k} (p_i(k) - v_i(k)),$$

where $p_i(k)$ and $v_i(k)$ are market price and promotion, respectively. The total force F_k is the sum of individual brand forces $f_i(k)$, which are evaluated at the prevailing market conditions.

- c Use steps (4a) and (4b) to create the starting vector $\mathbf{x}_0 = \text{vec}(y_{10}, y_{20}, \dots, y_{M0})$ of dimension $9 \times M$.

Step 5 Iteration

Initiate the iterations from \mathbf{x}_0 , apply step 3 and find the roots \mathbf{x}_1 . Then use \mathbf{x}_1 as the new starting values, and re-apply step 3; repeat until convergence.

This step yields the state and co-state trajectories for each brand.

Step 6 Optimal controls and performance index

- a Compute the fixed-point of the equations (12) and (16) using the state and co-state trajectories obtained in step 5. Specifically, for each brand i , start with

$$u_{i,0} = \lambda_i \alpha_i (1 - m_i) - b \quad \text{and} \quad v_{i,0} = \frac{\bar{v}_i \lambda_i \beta_i}{1 + \lambda_i \beta_i},$$

and then generate the Cauchy sequence $\{(u_{i,1}, v_{i,1}), \dots, (u_{i,r}, v_{i,r}), \dots, (u_i^*, v_i^*)\}$, where

$$u_{i,r+1} = \lambda_i (\alpha_i + \gamma_i v_{i,r}) (1 - m_i) - b \quad \text{and} \quad v_{i,r+1} = \frac{\bar{v}_i \lambda_i (\beta_i + \gamma_i u_{i,r})}{1 + \lambda_i (\beta_i + \gamma_i u_{i,r})}.$$

Repeat these steps for each $k = 1, \dots, M$.

- b Evaluate the switching function $D_{ik}^* = -m_{ik} + \lambda_{ik} (1 - m_{ik}) (\beta_i + \gamma_i u_{ik}^*)$ using the advertising policies from step (6a), and the state & co-state trajectories from step (3). Then, using the equation (15), obtain the bang-bang promotion policy for each brand.
- c Compute the performance index for the equilibrium policies in (6a, b):

$$J_i = \sum_{k=1}^M \frac{[(p_{ik} - v_{ik}) m_{ik} - c(u_{ik})]}{(1 + \rho)^k},$$

for each brand $i, i = 1, \dots, N$.

TABLE 4. ACTUAL VERSUS OPTIMAL: ADVERTISING, PROMOTION, AND PERFORMANCE

PANEL A. *Advertising Effort*

<i>Brand</i>	<i>Actual</i>	<i>Optimal</i>	<i>% Under-advertising</i>
Wisk	65.7	116.8	43.75
Tide	97.8	135.8	27.98
Bold	67.8	96.5	29.74
Era	86.6	96.3	10.07
Solo	80.2	94.1	14.77

PANEL B. *Promotion Effort*

<i>Brand</i>	<i>Actual</i>	<i>Optimal</i>	<i>Under-or Over-promotion</i>
Wisk	142.92	0	Over-promotion
Tide	167.48	0	Over-promotion
Bold	81.56	1.94	Over-promotion
Era	106.68	149.36	Under-promotion
Solo	115.54	167.34	Under-promotion

PANEL C. *Performance Index J*

<i>Brand</i>	<i>Actual</i>	<i>Optimal</i> <i>(with Interaction Effects)</i>	<i>Optimal</i> <i>(No Interaction Effects)</i>
Wisk	28.99	84.67	93.92
Tide	139.71	320.61	281.88
Bold	9.52	52.81	47.44
Era	17.71	43.87	44.98
Solo	18.76	42.94	44.33

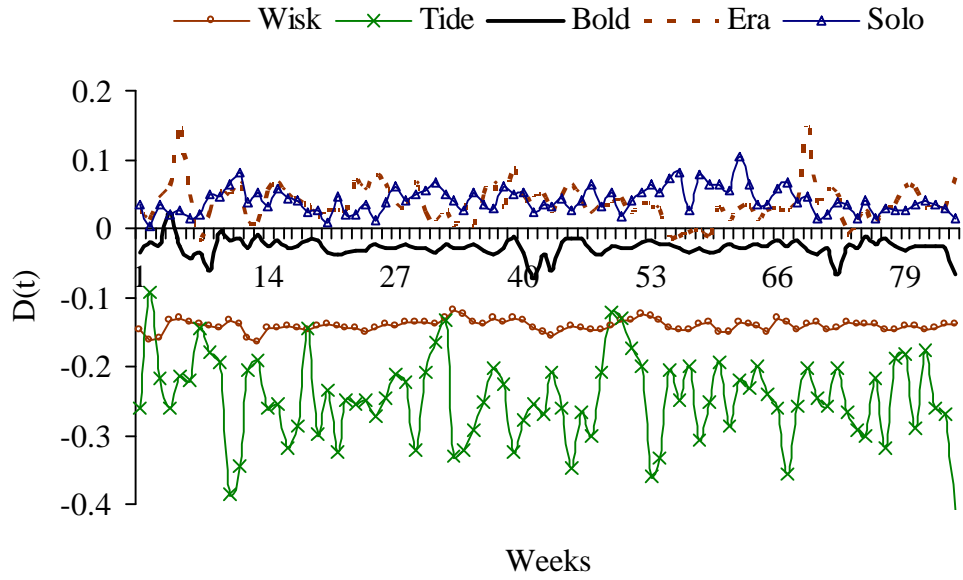
TABLE 5. COMPARATIVE STATICS FOR INTERACTION EFFECTS

	<i>Tide</i> ($\hat{\gamma} = -0.0954$) \uparrow		<i>Wisk</i> ($\hat{\gamma} = -0.0164$) \uparrow		<i>Era</i> ($\hat{\gamma} = 0.0032$) \uparrow		<i>Solo</i> ($\hat{\gamma} = -0.0127$) \uparrow		<i>Bold</i> ($\hat{\gamma} = -0.0203$) \uparrow	
	<i>Relative Allocation^a</i>	<i>Performance Impact^b</i>	<i>Relative Allocation</i>	<i>Performance Impact</i>	<i>Relative Allocation</i>	<i>Performance Impact</i>	<i>Relative Allocation</i>	<i>Performance Impact</i>	<i>Relative Allocation</i>	<i>Performance Impact</i>
<i>Tide</i>	Decrease	Positive	Decrease	Positive	Increase	Negative	Decrease	Positive	Decrease	Positive
<i>Wisk</i>	Decrease	Negative	Increase	Negative	Decrease	Negative	Increase	Positive	Increase	Positive
<i>Era</i>	Decrease	Negative	Increase	Positive	Increase	Positive	Increase	Positive	Increase	Positive
<i>Solo</i>	Decrease	Negative	Increase	Positive	Decrease	Negative	Decrease	Negative	Increase	Positive
<i>Bold</i>	Decrease	Negative	Increase	Positive	Decrease	Negative	Increase	Positive	Increase	Negative

Notes: (a) indicates change in advertising relative to promotion due to a marginal increase in interaction effect; (b) reveals profit impact due to a marginal increase in interaction effect.

FIGURE 1. SWITCHING FUNCTIONS

PANEL A. *Presence of Interaction Effects*



PANEL B. *Absence of Interaction Effects*

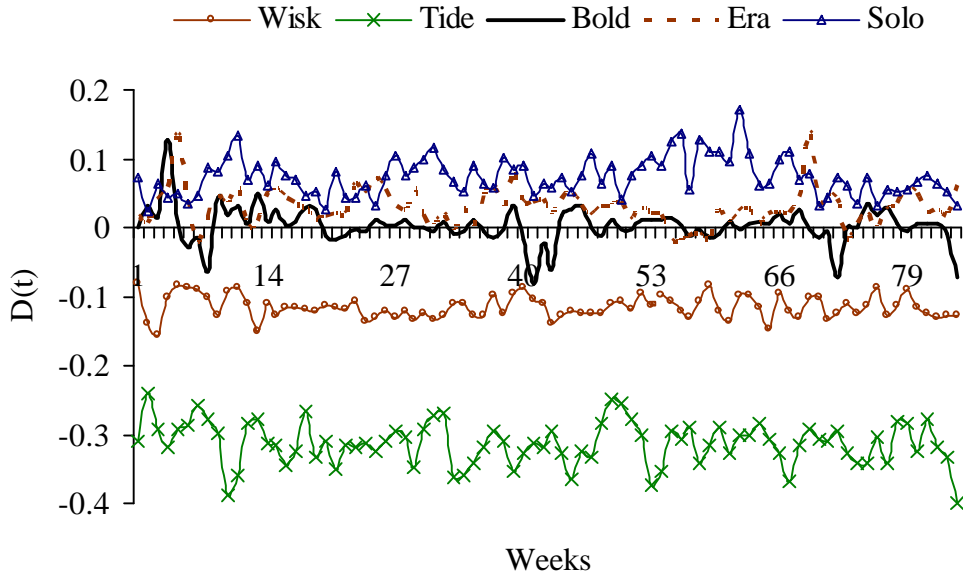
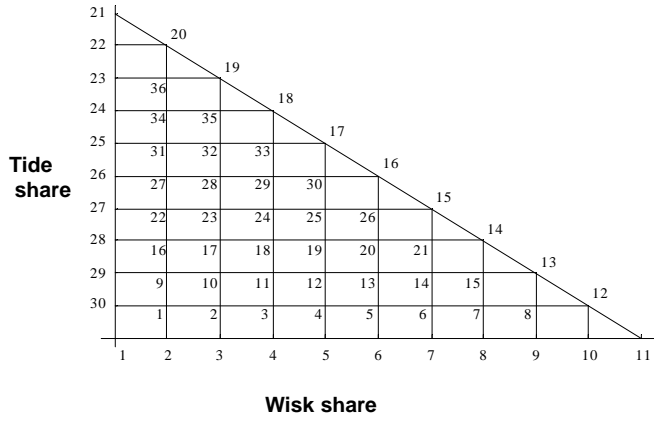
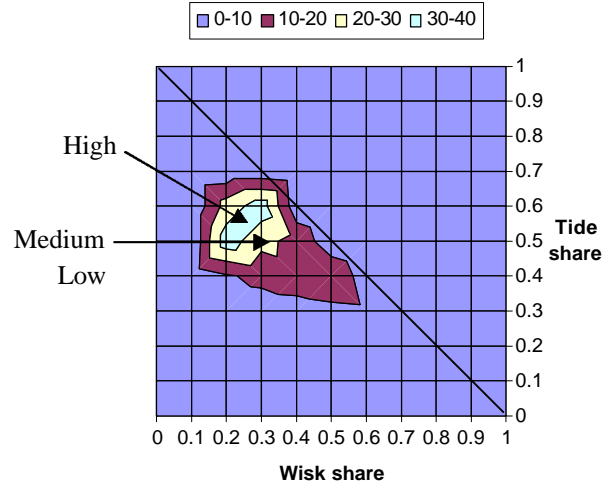


FIGURE 2. CLOSED-LOOP STRATEGIES IN TRIOPOLY

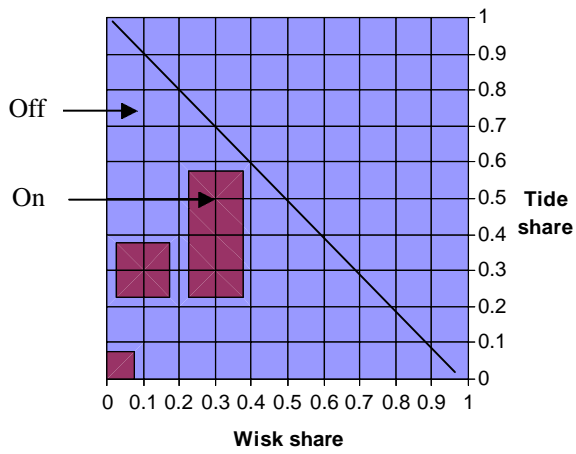
PANEL A. Domain of competition



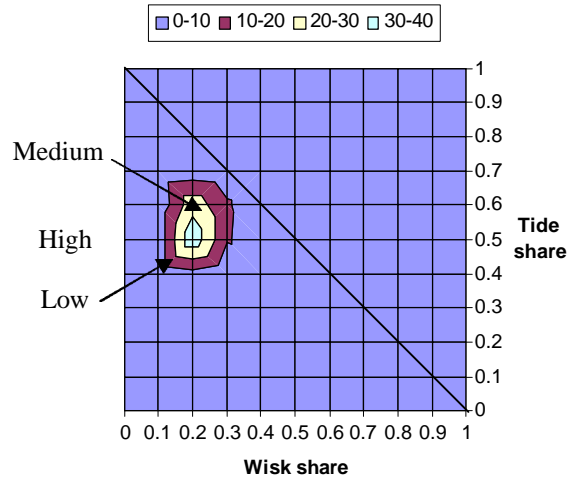
PANEL B. Contours of Wisk value function



PANEL D. Wisk promotion strategy



PANEL C. Wisk advertising strategy



APPENDIX A. TESTING EXOGENEITY

Here we test the hypotheses whether advertising and promotion are exogenous or not. For weak-exogeneity, we apply the test in Engle et al. (1983, 289) by estimating the marginal models of advertising and of promotion for each brand. Specifically, for Wisk, we obtain the following regression coefficients and t-values (in parentheses)

$$\hat{x}_{1k} = 0.3643 + 0.3777x_{1,k-1} + 0.6883y_{1,k-1},$$

(1.74) (2.86) (0.57)

where x_1 denotes brand advertising and y_1 is brand share. Similarly, using x_2 to denote Wisk's promotion, we estimate the regression,

$$\hat{x}_{2k} = 1.4748 - 0.0387x_{2,k-1} + 1.5026y_{1,k-1}.$$

(5.28) (-0.34) (1.48)

Then, we compute the residuals $(x_{1k} - \hat{x}_{1k})$ and $(x_{2k} - \hat{x}_{2k})$, and determine their correlations with the residuals from the conditional model, $(y_{1k} - \hat{y}_{1k})$. The resulting correlations (t-values in parenthesis) are 0.0766 (0.69) and -0.0904 (-0.82). Because the absolute t-values are less than 1.96, these correlations are not significant at the 95% confidence level, and hence Wisk advertising and promotion are weakly exogenous.

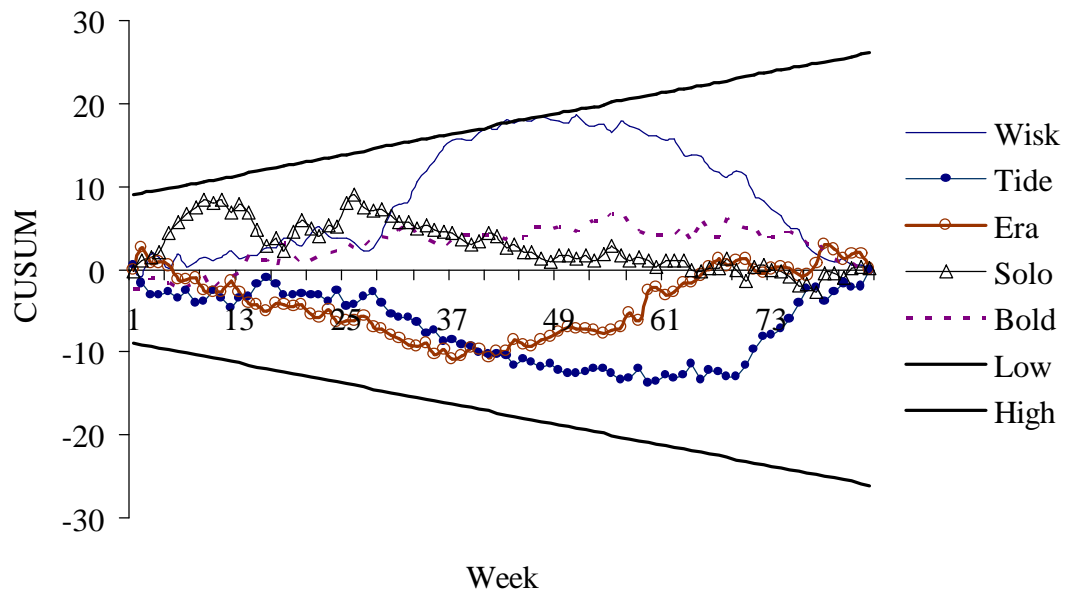
Similarly, we compute the correlations between residuals from conditional and marginal models for the other brands. For Tide advertising and promotion, respectively, the correlations (t-values in parenthesis) are 0.0243 (0.22) and -0.1214 (-1.10). For Era's advertising and promotion, the correlations are -0.0265 (-0.24) and 0.0357 (0.32). For Solo, these correlations are -0.0356 (-0.32) and -0.1877 (-1.72). The correlations for Bold's advertising and promotion are 0.0458 (0.41) and -0.1356 (-1.23). Based on these results, we observe that the absolute t-values are less than 1.96 (i.e., statistically insignificant at the 95% confidence level), implying weak-exogeneity of both advertising and promotion for every brand. Hence, the results in the manuscript are valid in the sense of efficient estimation (Engle et al. 1983, p. 290).

For super-exogeneity, we test parametric invariance via the CUSUM test developed by Brown et al. (1975). Specifically, we plot the cumulative sum of residuals, $CUSUM(t)$, as a function of time. The CUSUM statistic is defined as

$$CUSUM_i(t) = \hat{\sigma}_i^{-1} \sum_{k=1}^t \hat{\epsilon}_{ik},$$

where $\hat{\epsilon}_{ik} = (y_{ik} - za_{i,k|k-1}) / \sqrt{f_{ik}}$, $f_{ik} = \text{Var}(y_{ik})$, $\hat{\sigma}_i^2 = (T-1)^{-1} \sum_{k=1}^T (\hat{\epsilon}_{ik} - \bar{\hat{\epsilon}}_i)^2$, $\bar{\hat{\epsilon}}_i = T^{-1} \sum_{k=1}^T \hat{\epsilon}_{ik}$ for

brands $i = 1, \dots, N$, and $k = 1, \dots, T$. When $CUSUM_i(t)$ lies within the significance lines given by $\pm [a\sqrt{T} + 2at/\sqrt{T}]$, where $a = 0.948$ for the 95% confidence level, model parameters are deemed stable over time.



The above plot indicates that $CUSUM_i(t)$ is within the 95% confidence band for every brand. This finding, together with weak-exogeneity, results in super-exogeneity of advertising and promotion.

Based on above tests, we conclude that both advertising and promotion are weakly exogenous for every brand. In addition, both advertising and promotion are super-exogenous for every brand. Hence, the estimated model is adequate for efficient estimation (due to weak-exogeneity) and policy simulation (due to super-exogeneity) across all the five brands.